



The Müller–Breslau passive earth pressure coefficient: validity domain, degeneracy and quantification of the planar failure-surface error

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Abstract. The Müller–Breslau closed-form coefficient $K_p(\varphi, \alpha, \beta, \delta)$ is the standard design tool for cohesionless backfill, yet it overestimates passive resistance severely at high wall friction, and its validity domain has never been mapped systematically. Direct numerical minimization of the Coulomb trial wedge reproduces the closed form to machine precision over 45948 admissible combinations ($\varphi = 20\text{--}50^\circ$), so its difference from Sokolovski's method of characteristics isolates the planar failure-surface error. The closed form diverges exactly on the locus $\varphi + \delta + \beta - \alpha = 90^\circ$, where the admissible fan of trial planes closes, although the exact resistance stays finite. The error is unsafe in 99.6% of cells and scales with the margin $\varepsilon = 90^\circ - (\varphi + \delta + \beta - \alpha)$: median overestimation grows from about 2% for $\varepsilon > 50^\circ$ to over 50% for $\varepsilon < 20^\circ$, reaching factors of 1.5–5 at $\delta = \varphi$. Log-spiral, lower-bound, and finite-element limit-analysis bracketing (330 cases, 92.4% within $\pm 1\%$) and the Kérisel–Absi tables confirm this. The results are condensed into applicability maps with safe, caution, significant-error, and degenerate zones.

Keywords: passive earth pressure, Müller–Breslau formula, Coulomb wedge, method of characteristics, retaining wall, limit equilibrium

Introduction

The determination of passive earth resistance is a classical problem of soil mechanics whose most influential formulation remains the wedge solution of Coulomb [1]. Its generalization to an inclined wall, sloping backfill and wall friction, given in closed form by Müller-Breslau [2], is the standard closed-form design expression for cohesionless backfill: it appears in identical or lightly modified form in textbooks and numerous national standards, and — through the Mononobe-Okabe extension — in Annex E of Eurocode 8, while Annex C of Eurocode 7 replaces it for the passive case by charts and procedures based on curved slip surfaces — an implicit acknowledgement of the planar-surface error that this study quantifies. The formula expresses the passive earth pressure coefficient K_p as an explicit function of four angles — the soil friction angle φ , the wall inclination α , the backfill slope β and the wall friction angle δ — obtained analytically from the stationarity condition of the resistance with respect to the inclination ρ of a planar trial failure surface.

Two distinct questions accompany any closed-form limit-equilibrium solution: over what parameter domain is the expression valid, and how large is the error introduced by its mechanical idealization? For the passive Müller-Breslau formula, both questions are considerably more acute than for its active counterpart. The planar assumption is known to overestimate passive resistance — an error on the unsafe side wherever the passive resistance is relied upon — and textbooks accordingly warn against the formula for wall friction beyond roughly $\delta > \varphi/3$, recommending curved-surface tables instead [3], [4], [7]. The warning, however, is qualitative: no systematic map exists showing where in the $(\varphi, \alpha, \beta, \delta)$ space the formula is acceptable, how fast it degrades, and where it fails outright. Such outright failure indeed occurs: unlike the active coefficient, which remains finite over its entire algebraic domain, the passive coefficient grows without bound ($K_p \rightarrow \infty$) as the square root in the denominator of (1) approaches unity. This degeneracy has no counterpart in the active case; it is usually mentioned only as an algebraic curiosity and has, to the authors' knowledge, never been mapped, explained mechanically, or connected to the behaviour of the underlying wedge mechanism.

The error of the planar failure surface in the passive case has a long history. Rigorous reference solutions are provided by the method of characteristics of Sokolovski [5], whose framework continues to be applied to passive earth pressure problems [14], [19], and by the log-spiral tables of Caquot and Kérisel [6] and Kérisel and Absi [7], which remain the reference of record in design practice. Within limit analysis [8], kinematic (upper-bound) solutions with translational multiblock and rotational log-spiral mechanisms [9], [10], [24] and static (lower-bound) stress fields — most notably the closed-form solution of Lancellotta [11] — bracket the exact collapse load, and modern finite-element limit analysis provides tight numerical bounds for arbitrary geometries [13]. The slip-line framework has been extended to critical slip fields for both limit states [15], general wall and backfill geometry [16], [17], c - φ and seismic conditions [18], [19], and axisymmetric and kinematic variants [20]–[23]. From these comparisons, it is generally accepted that Coulomb-type solutions may overestimate passive resistance drastically — by a factor of two to three at $\delta = \varphi$ [7], [8]. However, published comparisons typically report differences

between complete computational chains — different numerical methods, discretizations and sign conventions — so that the contribution of the planar surface itself is not cleanly isolated; and the divergence of the closed form is treated as a boundary of the tables rather than as a phenomenon with a mechanical meaning of its own.

The present paper addresses both questions in the unified computational framework developed for the active case in a companion study [25]. First, the admissible domain of the passive Müller-Breslau formula is mapped over the practical range $\varphi = 20\text{--}50^\circ$, covering the full physical ranges of α , β and δ , and the degeneracy is characterized analytically: a trigonometric identity is derived showing that the radicand reaches unity exactly on the locus $\varphi + \delta + \beta - \alpha = 90^\circ$, which is simultaneously the configuration at which the singular direction of the trial-wedge problem reaches the backfill surface and the admissible fan of trial planes closes — the divergence of the formula and the disappearance of the wedge mechanism are one and the same event. Second, the closed form is verified against direct numerical minimization of the Coulomb trial wedge, formulated with two admissibility conditions that are seldom stated explicitly: the geometric bound $\rho > \beta$ and the singular-direction bound $\rho < \rho_{\text{sing}} = 90^\circ - \varphi - \delta + \alpha$, beyond which the force polygon admits only non-physical solutions. The agreement to machine precision over all 45 948 combinations of the parametric $(\varphi, \alpha, \beta, \delta)$ grid defined in the Methodology establishes that the difference between the Müller-Breslau solution and the method of characteristics is a pure measure of the planar failure-surface error. This error is then quantified over the whole parameter space with a validated passive slip-line solver, cross-checked by an independent rotational log-spiral upper-bound mechanism and by the static lower bound of Lancellotta — so that the exact coefficient is bracketed from both sides — and anchored to the Kérisel-Absi tables. The results are condensed into applicability maps that divide the parameter space into zones of safe application, caution, significant error, and the degenerate zone in which the closed form yields no prediction although the exact resistance remains finite. All results can additionally be explored through interactive web applications provided as supplementary material.

The methodology

The Müller-Breslau closed form and sign conventions

The geometry and sign conventions used throughout this paper are defined in Figure 1 and are identical to those of the companion active-case study [25], with one difference dictated by the passive kinematics: the wall friction angle δ is positive when the soil rises relative to the wall, as it does when passive resistance is mobilized. The wall face is inclined at α to the vertical (positive when the soil rests on the inclined face), and the backfill surface at β to the horizontal (positive upward away from the wall). The passive earth pressure coefficient is given by Müller-Breslau [2] as

$$K_p = \cos^2(\varphi + \alpha) / \{ \cos^2\alpha \cdot \cos(\alpha - \delta) \cdot [1 - \sqrt{(\sin(\varphi + \delta) \cdot \sin(\varphi + \beta) / (\cos(\alpha - \delta) \cdot \cos(\alpha - \beta)))^2} \} \quad (1)$$

and defines the total passive resistance per unit length of wall,

$$P_p = \frac{1}{2} \cdot K_p \cdot \gamma \cdot H^2 \quad (2)$$

where γ is the unit weight of the backfill and H the vertical height of the wall; the factor $\cos^2 \alpha$ accounts for the increased length of the inclined face. The resultant acts at the angle δ to the normal of the wall face, on the opposite side of the normal than in the active case, so that the horizontal and vertical components follow as

$$P_h = P_p \cdot \cos(\delta - \alpha), \quad P_v = P_p \cdot \sin(\delta - \alpha) \quad (3)$$

For $\alpha = 0$, these reduce to the familiar $\cos \delta$ and $\sin \delta$ decomposition. Eq. (3) gives the components of the resultant exerted by the soil on the wall, with positive P_v acting upward along the face for $\delta > \alpha$; the wall exerts an equal and opposite force on the soil wedge (Figure 1) — the sign convention affects P_v only, the cosine in P_h being an even function. For inclined walls, the use of $\cos \delta$ alone is a common source of error, as is the confusion between K_p and the horizontal coefficient $K_{ph} = K_p \cdot \cos(\delta - \alpha)$ tabulated by some references. Expression (1) is real-valued for $\beta \geq -\varphi$ and $\delta \geq -\varphi$, with the additional requirements $\alpha - \delta < 90^\circ$ and $|\alpha - \beta| < 90^\circ$ for the cosines in the denominator to remain positive. Unlike the active coefficient, however, the passive expression possesses an interior singularity: when the radicand approaches unity, the denominator of (1) vanishes and $K_p \rightarrow \infty$. The location and mechanical meaning of this degeneracy are established in the Results.

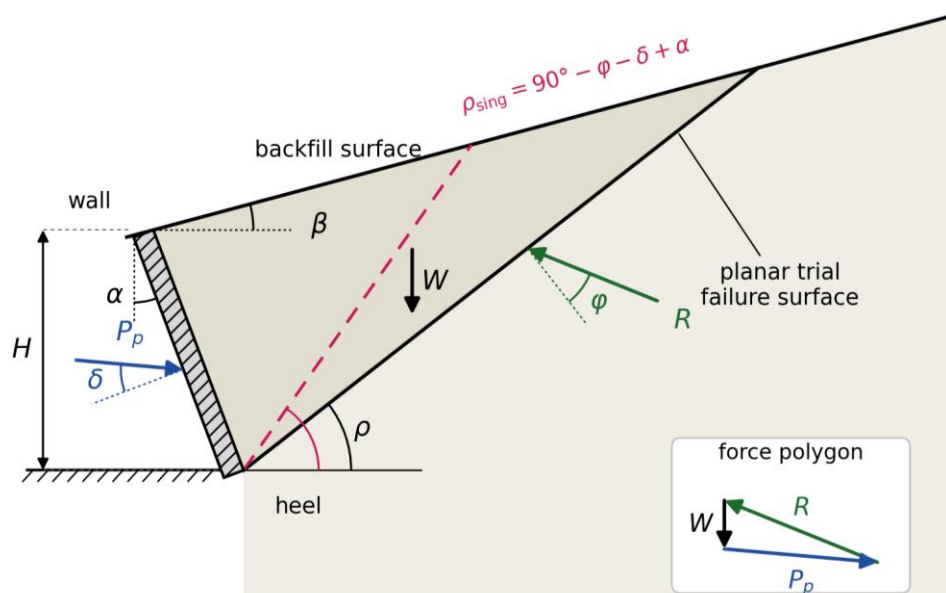


Figure 1. Definition of the geometry, sign conventions and forces of the passive Coulomb trial-wedge problem. All angles are shown in their positive directions; the trial-plane inclination ρ is measured from the horizontal at the wall heel, with the admissible range $\beta < \rho < \rho_{\text{sing}}$; the singular direction $\rho_{\text{sing}} = 90^\circ - \varphi - \delta + \alpha$ is marked in red. The force polygon is drawn for the wedge: the wall transmits to the wedge a force equal and opposite to the passive resultant P_p .

Direct numerical minimization of the Coulomb wedge

Independently of (1), the passive resistance was computed by direct numerical minimization of the Coulomb trial-wedge problem. For a planar trial surface inclined at ρ to the horizontal and passing through the wall heel (Figure 1), the equilibrium of the wedge under its weight W , the reaction R on the slip plane (inclined at φ to the plane's normal, opposing the upward motion of

the wedge) and the thrust transmitted by the wall (inclined at δ to the face normal, equal and opposite to the passive resultant P_p) yields the resistance $E(\rho)$ in closed form from the force polygon; the passive resistance is $P_p = \min E(\rho)$. Written as a coefficient, the trial-wedge resistance takes the compact form

$$K_p(\rho) = \cos(\alpha - \beta) \cdot \cos(\rho - \alpha) \cdot \sin(\rho + \varphi) / [\cos^2 \alpha \cdot \sin(\rho - \beta) \cdot \cos(\rho + \varphi + \delta - \alpha)] \quad (4)$$

Two admissibility conditions, seldom stated explicitly in the literature, are essential for a robust numerical treatment — and far more restrictive than their active counterparts. First, the trial plane must emerge above the backfill surface, $\rho > \beta$; as $\rho \rightarrow \beta$ the wedge grows without bound and (4) diverges. Second, the trial plane must remain below the singular direction,

$$\rho < \rho_{\text{sing}} = 90^\circ - \varphi - \delta + \alpha \quad (5)$$

at which the wall and slip-plane reactions become parallel: the last cosine in the denominator of (4) vanishes, the force polygon degenerates, and for $\rho > \rho_{\text{sing}}$ equilibrium admits only non-physical solutions with negative force magnitudes. In the active case, the singular direction lies at the outer boundary of the parameter space and merely requires the compression check $R \geq 0$; in the passive case, the fan ($\beta, \rho_{\text{sing}}$) of admissible trial planes narrows rapidly with growing φ, δ and β and with decreasing α , and its width practically defines the usable domain of the wedge model. Since the objective function of the minimization, $K_p(\rho)$ in (4), is defined only on the fan and is infinite outside it, standard derivative-free bracketing minimizers (golden-section search with successive parabolic interpolation, i.e. the Brent family) fail on the resulting plateau; the minimization was therefore carried out by a dense uniform scan of the admissible fan followed by parabolic refinement, which resolves the minimum to machine precision (verified against an independent implementation to within $5 \cdot 10^{-15}$ relative on a 26-point control table). One further implementation detail deserves emphasis: for $\beta < 0$ near the degeneracy locus, the critical inclination itself becomes negative — the critical plane dips below the horizontal (for example, $\rho_{\text{cr}} = -4.8^\circ$ at $\varphi = 45^\circ, \alpha = 0^\circ, \beta = -10^\circ, \delta = 44^\circ$) — so that a search restricted to $\rho > 0$, as is natural in the active case, silently returns a wrong optimum, in the quoted example 232% above the true one.

Reference solution: method of characteristics

Rigorous reference values were computed with a slip-line solver implementing Sokolovski's method of characteristics for the passive limit state of a cohesionless soil with self-weight [5]. The governing hyperbolic system is integrated along the two characteristic families, marching from the stress boundary condition on the backfill surface toward the wall, with the free parameter iteratively adjusted until the wall friction condition is satisfied. Depending on the parameter combination, the solution takes one of three structures, illustrated in the companion active-case study [25] and in Sokolovski's monograph [5]: (A) a continuous field; (B) a field containing a stress discontinuity; or (C) a degenerate single-zone Rankine-type field when the wall condition is fully consistent with the surface condition. On the parametric grid described below (56 364 cells, of which 770 with $|\beta| \geq \varphi$ are physically inadmissible and 13 did not converge, see the Parametric study subsection), the 55 581 converged solutions divide into 46 339 of structure A, 9 194 of structure B and 48 of structure C; the structure-C cells are confined to the corner of positive wall

inclination with nearly full wall friction ($\alpha \geq +10^\circ$, $\delta/\varphi \geq 5/6$), and it is precisely there that the planar-surface error nearly vanishes (median -0.3%).

The solver was validated in six independent ways. First, for the frictionless vertical wall with horizontal backfill, it reproduces the Rankine value $K_p = \tan^2(45^\circ + \varphi/2)$ to within 0.036% over $\varphi = 20\text{--}50^\circ$. Second, its results respect the kinematic theorem: all 35 tabulated upper-bound coefficients of Soubra [24] (translational multiblock mechanism, $\varphi = 15\text{--}45^\circ$, $\delta/\varphi = 0\text{--}1$, vertical wall, horizontal backfill) bound the present solutions from above, with the ratio $K_{p,\text{SOK}}/K_{p,\text{UB}}$ decreasing smoothly from 1.000 at $\delta = 0$ (both methods exact) to 0.904 at $\varphi = 45^\circ$, $\delta = \varphi$ — consistent with the roughly 12% gap between that mechanism and the Caquot-Kérisel tables reported by Soubra himself. Third, at the classical anchor point $\varphi = 30^\circ$, $\alpha = \beta = 0$, $\delta \rightarrow \varphi$, the solver yields $K_p = 6.54$, against approximately 6.55 read from the Kérisel-Absi tables [7]. Fourth, as reported in the next subsection, the solution is bracketed from both sides by an independent upper and lower bound at every verified point. Fifth, the entire rough-wall branch $\delta = \varphi$ was compared with the Kérisel-Absi log-spiral tables [7] over all six friction angles and eleven wall inclinations (66 combinations, converted between normalisations by $K_p = K_{p,K-A}/\cos^2\alpha$): the median difference is -0.1% , and all readings agree within $\pm 2\%$ except a single one at the two-digit resolution limit of the table. Sixth, as described in the following subsection and in the Results, the solution was bracketed by rigorous finite-element limit-analysis bounds over a 330-case verification matrix. Direct comparison with Sokolovski's own tabulated passive examples is not possible without extension of the present framework, since those examples include a surface surcharge [5]; this extension is left for future work. The mobilized wall friction is capped at the value consistent with the characteristic field ($\delta_{\text{mob}} \leq \delta$); the cap was active in only 48 cells of the grid, all flagged — precisely the 48 structure-C cells.

Independent verification: rotational log-spiral upper bound and a static lower bound

As an independent cross-check of the characteristics solution, the passive resistance was additionally bracketed from both sides. From above, a rotational log-spiral mechanism was constructed in the framework of the kinematic theorem of limit analysis [8], [10], mirroring the active-case implementation [25]: a rigid region bounded by the wall face, the backfill surface and a log-spiral slip surface through the wall heel rotates about a center whose two coordinates are the optimization variables; the spiral pitch equals $\tan \varphi$, so that the velocity jump is inclined at φ to the surface and no internal energy is dissipated in the cohesionless soil. The resistance follows from the balance of the work rates of gravity and of the wall reaction, assumed to act at the lower third of the face at the angle δ to its normal, and is minimized over the position of the rotation center. Since the planar wedge is the limiting member of this family as the center recedes to infinity, the mechanism yields $K_{p,\text{LS}} \leq K_{p,\text{MB}}$, while the kinematic theorem guarantees $K_{p,\text{LS}} \geq K_{p,\text{exact}}$ for the passive problem; the mechanism therefore brackets the exact coefficient from above while undercutting the planar solution, and provides a one-sided check on the characteristics solution that does not depend on its implementation. The implementation reproduces the Rankine value to within 0.01% in the smooth-vertical-wall limit. From below, the closed-form static solution of Lancellotta [11], derived from a statically admissible stress field with a stress-fan rotation between

the backfill and the wall, provides a rigorous lower bound for the vertical wall with horizontal backfill:

$$K_{p, LB} = [\cos \delta + \sqrt{(\sin^2 \varphi - \sin^2 \delta)}] / (1 - \sin \varphi) \cdot \exp(2\theta \cdot \tan \varphi), \quad 2\theta = \delta + \arcsin(\sin \delta / \sin \varphi) \quad (6)$$

written here for the resultant coefficient. Together, the two bounds enclose the exact coefficient in the sandwich $K_{p, LB} \leq K_{p, exact} \leq K_{p, LS}$, with the characteristics solution expected to fall inside the bracket and the Müller-Breslau value above its upper end wherever the planar error is significant. This bracketing, verified numerically at every control point in the Results, is the passive-case counterpart of the ordering $MB \leq LS \leq SOK$ observed in the active study — with the direction of every inequality reversed.

Verification by finite-element limit analysis

Finally, the slip-line solution was verified against independent finite-element limit analyses (FELA) performed with OPTUM G2 [27]. FELA computes rigorous lower-bound (LB) and upper-bound (UB) estimates of the collapse load for an associated Mohr-Coulomb material; since the slip-line solution addresses the same idealised problem (associated flow, rigid wall, full mobilisation of wall friction), the exact coefficient must satisfy $K_{p, LB} \leq K_{p, SOK} \leq K_{p, UB}$ at every point. The verification matrix comprised the full factorial grid of six friction angles $\varphi = 20\text{--}45^\circ$ (step 5°), five friction ratios $\delta/\varphi \in \{0, 1/3, 1/2, 2/3, 1\}$ and eleven wall inclinations $\alpha = -25\text{...}+25^\circ$ (step 5°) at horizontal backfill ($\beta = 0$) — $6 \times 5 \times 11 = 330$ combinations — with $\gamma = 20 \text{ kN/m}^3$ and $H = 1 \text{ m}$, deliberately including the seventeen combinations beyond the degeneracy locus, where the closed form yields no prediction at all.

Each case was modelled as a rigid, weightless block adjoining the soil, with the wall face represented by a rigid plate and a Mohr-Coulomb interface whose strength-reduction factor was set to $r = \delta/\varphi$; the block was restrained against vertical translation and rotation and pushed horizontally by a unit multiplier load, so that the collapse multiplier equals the horizontal component $P_h = P_p \cdot \cos(\delta - \alpha)$ of the passive resultant. The wall-modelling approach follows Schmüdderich et al. [26]. Both bounds used 10 000 target elements with five (near the degeneracy locus, eight) cycles of adaptive mesh refinement under shear-dissipation control and mesh fans at the wall heel and crest. A prescribed-traction formulation (a distributed load applied directly to the soil boundary, inclined at δ to the face normal) served as a cross-check for $\delta/\varphi \leq 2/3$; for $\delta = \varphi$, that formulation places the boundary traction exactly on the Mohr-Coulomb yield cone and the lower bound degenerates, which is why the rigid-block model was adopted throughout. The outcome of the 330-case programme is reported with the other verifications in the Results.

Parametric study and applicability classification

The parametric grid covers φ from 20° to 50° in steps of 1° , $\alpha \in \{-25, -20, \dots, +25\}^\circ$, β from -25° to $\varphi - 2.5^\circ$ in steps of 2.5° , and δ from 0 to φ in steps of $\varphi/6$ — in total 56 364 combinations. Cells with $|\beta| \geq \varphi$ (770) violate the natural stability limit of a cohesionless slope and are excluded as physically inadmissible regardless of the calculation method. The corner $\delta = \varphi$, at which the radicand of (1) is evaluated on the boundary of its domain, is computed at $\delta = \varphi - 0.05^\circ$ and flagged (8 052 cells). Cells on or beyond the degeneracy locus, $\varepsilon = 90^\circ - (\varphi + \delta + \beta - \alpha) \leq 0$, at which the

closed form yields no prediction, are flagged as the degenerate zone D (9 646 cells) — and the characteristics solver is run there regardless, successfully in 9 645 of them. Overall, 55 581 cells (98.61%) yielded a converged characteristic solution; of the non-converged remainder, only 13 cells (0.02% of the grid) represent genuine non-convergence, 11 of them on the steepest admissible backfill line and two isolated cells at high wall friction. For each cell with a defined closed-form value, the relative difference $\Delta = (K_{p,SOK} - K_{p,MB})/K_{p,MB}$ is evaluated, and the parameter space is classified into zone A, $|\Delta| < 2\%$, in which the planar-surface error is below the practical uncertainty of determining φ itself; zone B, $2\% \leq |\Delta| < 5\%$, comparable to the margins implied by partial safety factors; zone C, $|\Delta| \geq 5\%$; and zone D, the degenerate region without a closed-form prediction. The thresholds are identical to those of the active study [25], so that the two maps are directly comparable. Interactive web applications through which the complete parameter space can be explored are provided as supplementary material.

Findings/Discussion

Equivalence of the closed form and the numerical wedge optimum

Over all 45 948 grid cells at which the closed form is defined, expression (1) and the direct numerical minimization of the constrained trial wedge (4) agree to within $10^{-10} \%$ — machine precision (maximum relative deviation $9.9 \cdot 10^{-11} \%$). No parameter combination was found within the physical domain at which the constrained minimum falls on a boundary of the admissible fan $(\beta, \rho_{\text{sing}})$ rather than at the interior stationary point; the closed form and the constrained numerical optimum are therefore computationally interchangeable, and the degeneracy of the Müller-Breslau solution occurs exclusively on the degeneracy locus, not inside the domain. This equivalence is the key that unlocks the remainder of the paper: since the closed form is an exact evaluation of the planar-wedge model, any difference between it and a rigorous solution measures the planar-surface assumption alone. The behaviour of K_p over the (β, δ) plane for successive wall inclinations is shown in Figure 2, in which the degeneracy locus appears as the straight line $\delta = 90^\circ + \alpha - \beta - \varphi$; the assembled three-parameter function $K_p(\alpha, \beta, \delta)$ can be explored in the interactive applications provided as supplementary material.

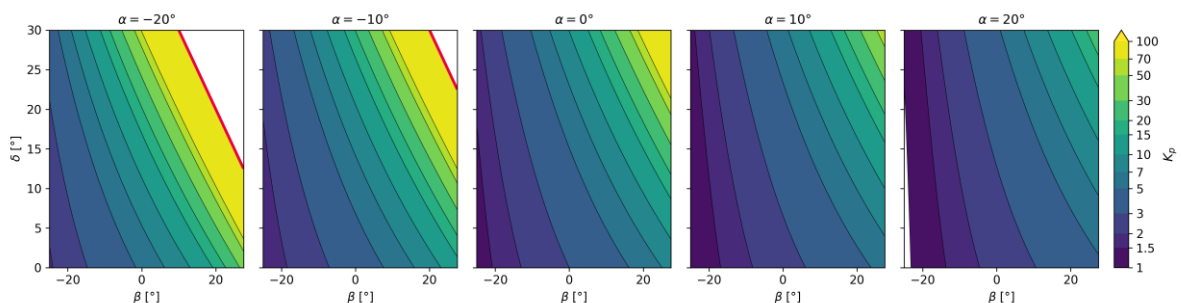


Figure 2. Müller-Breslau coefficient $K_p(\beta, \delta)$ at constant wall inclination α (example shown for $\varphi = 30^\circ$), with the degeneracy locus $\varphi + \delta + \beta - \alpha = 90^\circ$ (red line) and the slope-stability boundary $\beta = \varphi$ (dashed) marked.

The singular direction, the critical plane, and the degeneracy $K_p \rightarrow \infty$ The location of the degeneracy admits an exact characterization. Substituting the singular direction (5) into

elementary trigonometric identities yields

$$\sin(\varphi + \delta) \cdot \sin(\varphi + \beta) - \cos(\alpha - \delta) \cdot \cos(\alpha - \beta) = -\cos(\varphi + \alpha) \cdot \cos(\varphi + \delta + \beta - \alpha) \quad (7)$$

so that the radicand of (1) equals unity exactly when $\cos(\varphi + \delta + \beta - \alpha) = 0$, i.e. on the locus $\varphi + \delta + \beta - \alpha = 90^\circ$ (the factor $\cos(\varphi + \alpha)$ does not vanish in the physical domain). On the same locus, $\rho_{\text{sing}} = \beta$: the singular direction coincides with the backfill surface, and the admissible fan of trial planes $(\beta, \rho_{\text{sing}})$ closes to zero width. The divergence of the closed form and the disappearance of the wedge mechanism are therefore one and the same event — a coincidence with no counterpart in the active case, where the coefficient remains finite while the wedge merely grows unbounded at $\beta \rightarrow \varphi$. Identity (7) follows in three lines from the product-to-sum and sum-to-product expansions and was additionally verified numerically to machine precision. Near the locus the divergence follows the asymptotic law $K_p \propto \varepsilon^{-2}$, with

$$\sin^2 \varepsilon \cdot K_p \rightarrow 4 \cdot \cos(\alpha - \delta) \cdot \cos^2(\alpha - \beta) / \cos^2 \alpha \quad \text{as } \varepsilon \rightarrow 0 \quad (8)$$

where $\varepsilon = 90^\circ - (\varphi + \delta + \beta - \alpha)$ is the margin to the locus; the margin will prove to be the single parameter that organizes the entire error landscape. Figure 3 maps the critical plane inclination $\rho_{\text{cr}}(\beta, \delta)$: as the locus is approached the fan narrows, ρ_{cr} is driven toward the lower admissibility bound β , and for $\beta < 0$ the critical plane dips below the horizontal, as noted in the Methodology. Beyond the locus lies the degenerate zone D — 17.4% of the physically admissible grid — in which the formula yields no prediction at all while the exact resistance remains finite: the characteristics solver converged in 9 645 of its 9 646 cells (the single failure, $\varphi = 26^\circ$, $\alpha = -25^\circ$, $\beta = 22.5^\circ$, $\delta = 2\varphi/3$, combines a backfill slope within 3.5° of the natural-slope limit with the strongest overhang), with coefficients between 11.3 and 5 300 (median 81.4). At $\varphi = 50^\circ$, $\alpha = \beta = 0$, $\delta = \varphi$, for instance, the closed form has already diverged while the rigorous coefficient is 77.8 — and, as shown below, the exact value there is provably confined to the interval [50.5, 89.7] by the two bounds.

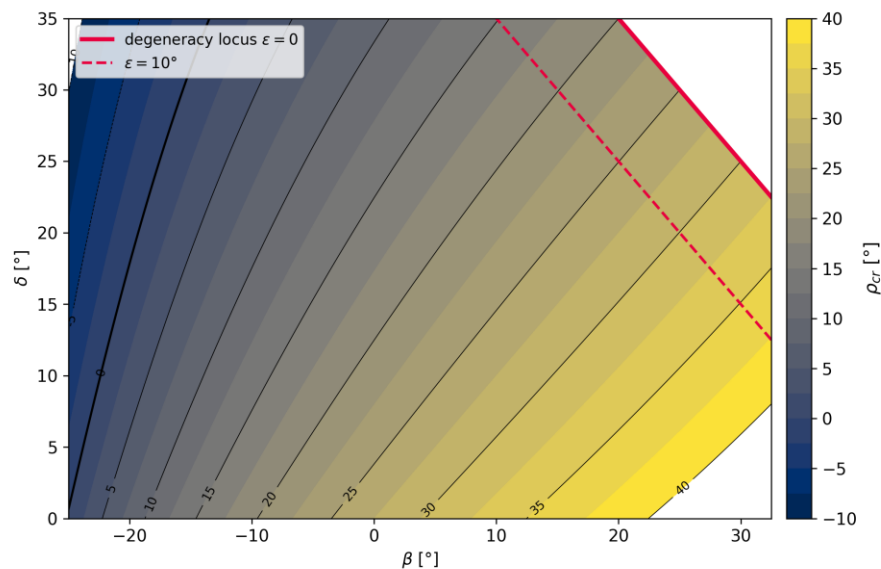


Figure 3. Critical failure-plane angle $\rho_{\text{cr}}(\beta, \delta)$ from the numerical wedge minimization (example shown for $\varphi = 35^\circ$, $\alpha = 0^\circ$). The full red line is the degeneracy locus $\varepsilon = 0$; the dashed red line marks the practical proximity contour $\varepsilon = 10^\circ$.

Validation by rigorous bracketing

Table 1 assembles, for the vertical wall with horizontal backfill, the four solutions available in this study: the Lancellotta lower bound (6), the method of characteristics, the rotational log-spiral upper bound, and the Müller-Breslau closed form. Three observations follow. First, at $\delta = 0$, all four coincide with the Rankine value to four digits — the planar surface is then exact, and the agreement validates the sign conventions of all implementations simultaneously. Second, the theoretical ordering $LB \leq SOK \leq LS \leq MB$ holds in every row without exception, as it does at every other point verified in this study; the characteristics solution always falls inside the rigorous bracket. Third, the bracket $[LB, LS]$ is narrow where the closed form is accurate and remains moderate (width 15–78% of the characteristic value) even at $\delta = \varphi$, where the closed form has drifted far above it or diverged entirely: at $\varphi = 50^\circ$, $\delta = \varphi$, the exact coefficient is confined to $[50.5, 89.7]$ while the planar theory offers no value at all.

Table 1. Bracketing validation for the vertical wall with horizontal backfill ($\alpha = \beta = 0$): Lancellotta lower bound, method of characteristics, rotational log-spiral upper bound and Müller-Breslau closed form. $\delta = \varphi$ evaluated at $\varphi - 0.05^\circ$; '—' marks the degenerate case without a closed-form value. Direct evaluation at $\delta = \varphi$ exactly yields different values; for $\varphi = 50^\circ$, $\delta = \varphi$ the closed form lies beyond the degeneracy locus, where blind evaluation returns a finite but spurious value.

No.	φ [°]	δ/φ	$K_{p,LB}$ (6)	$K_{p,SOK}$	$K_{p,LS}$	$K_{p,MB}$ (1)
1	20	0	2.040	2.039	2.039	2.040
2	20	1/2	2.515	2.559	2.568	2.635
3	20	1 ($\varphi - 0.05^\circ$)	2.871	3.043	3.126	3.519
4	30	0	3.000	3.000	3.000	3.000
5	30	1/2	4.439	4.614	4.649	4.977
6	30	1 ($\varphi - 0.05^\circ$)	5.801	6.543	6.920	10.066
7	40	0	4.599	4.599	4.599	4.599
8	40	1/2	8.916	9.667	9.815	11.771
9	40	1 ($\varphi - 0.05^\circ$)	14.382	18.112	19.981	91.704
10	45	0	5.828	5.828	5.828	5.828
11	45	1/2	13.602	15.281	15.615	21.379
12	45	1 ($\varphi - 0.05^\circ$)	25.449	34.897	39.412	$3.72 \cdot 10^6$
13	50	0	7.549	7.549	7.548	7.549
14	50	1/2	22.292	26.388	27.217	49.025
15	50	1 ($\varphi - 0.05^\circ$)	50.482	77.808	89.716	—

Planar-surface error across the parameter space

Table 2 gives the median of the relative difference $\Delta = (K_{p,SOK} - K_{p,MB})/K_{p,MB}$ over the (α, β) plane for each combination of φ and δ/φ ; the median is reported because the distribution is heavy-

tailed toward the degeneracy locus, so that the mean would be dominated by the divergent cells; the tail is characterized separately below. Two observations organize the error landscape. First, the difference is negative in 99.6% of all cells (maximum positive value +0.13%, within the numerical noise of the comparison): the method of characteristics yields the smaller coefficient, so the planar assumption errs on the unsafe side for passive resistance essentially everywhere — the exact mirror image of the active case, and with far larger magnitudes. (One qualified region exists: for $\delta = 0$ and $\alpha \geq +10^\circ$ the finite-element verification reported below shows the planar mechanism to be exact and the slip-line construction conservative, so the small differences there overstate the planar error; the $\delta = \varphi$ medians that dominate the practical conclusions are unaffected.) The overall median is -10.7% ; the median at full wall friction $\delta = \varphi$ is -33.6% , corresponding to a median overestimation factor $K_{p,MB}/K_{p,SOK}$ of 1.51, with the 90th percentile of that factor at 13 and, in the band adjoining the locus ($0 < \varepsilon \leq 2^\circ$), values diverging without bound (median Δ there: -99.6%). At the representative points $\varphi = 20, 30, 40^\circ$ ($\alpha = \beta = 0, \delta = \varphi$) the overestimation factors are 1.16, 1.54 and 5.06 respectively, and at $\varphi = 45^\circ$ the closed form has left the physically meaningful range altogether.

Second, no single input parameter governs the error. The medians of Table 2 grow with δ/φ (from -6.0% to -33.6% over the pooled grid) — but they grow comparably with the wall inclination (from -5.1% at $\alpha = +25^\circ$ to -36.6% at $\alpha = -25^\circ$), with the backfill slope, and with φ . All four trends collapse onto one when the error is plotted against the margin $\varepsilon = 90^\circ - (\varphi + \delta + \beta - \alpha)$ to the degeneracy locus, as summarized later in Figure 10a. How strongly ε governs the error can be stated in two ways. First, correlation: over the whole grid the Spearman rank correlation (a measure of monotone association) between $|\Delta|$ and ε is -0.83 , while no individual parameter exceeds 0.48 in absolute value (δ/φ : 0.27; α : -0.29 ; β : 0.48; φ : 0.22) — the error follows the combined margin, not any single angle. Second, the trend itself: the typical (median) error decreases steadily as the margin grows — from essentially 100% for $\varepsilon \leq 2^\circ$, through 52% at $\varepsilon = 15-20^\circ$, 13.5% at $\varepsilon = 30-40^\circ$ and 4% at $\varepsilon = 40-50^\circ$, to about 2% beyond $\varepsilon = 50^\circ$ — and within any given ε -band the spread between cells is small compared with this overall fall. The margin to the degeneracy — not the wall friction as such — is the governing parameter of the passive planar-surface error, the structural counterpart of the wall-inclination dominance found in the active case [25]. It also yields a one-line screening rule for practice: the closed form should not be trusted beyond the few-percent level unless ε exceeds roughly $45-50^\circ$.

Because the largest differences occur where any single numerical method might be suspected of degradation, the characteristics solution was cross-checked at control points spanning the parameter domain, including inclined walls and sloping backfill (Table 3), and along the full wall-friction range $0 \leq \delta \leq \varphi$ at fixed φ (Figure 4).

Table 2. Median relative difference Δ [%] between the method of characteristics and the Müller-Breslau solution, over all wall inclinations and backfill slopes of the grid. Negative values: the closed form overestimates the passive resistance. In the $\delta/\varphi = 0$ column a part of the difference at $\alpha > 0$ reflects conservatism of the slip-line construction rather than planar-surface error (family F1 of the finite-element verification).

Independent verification: bounds, Kérisel-Absi tables and finite-element limit analysis

φ [°]	$\delta/\varphi = 0$	1/6	1/3	1/2	2/3	5/6	1
20	-2.5	-2.6	-3.0	-3.5	-5.4	-8.6	-12.7
25	-3.6	-4.0	-4.4	-5.3	-8.8	-13.8	-20.6
30	-5.0	-5.5	-6.4	-7.5	-13.7	-20.8	-30.4
35	-6.1	-6.9	-8.4	-12.7	-21.0	-31.2	-43.3
40	-7.6	-8.7	-10.8	-17.5	-29.1	-39.8	-53.9
45	-9.0	-10.1	-12.3	-21.1	-32.1	-43.2	-60.0
50	-10.1	-11.6	-14.7	-22.9	-36.0	-49.6	-64.5

Table 3. Three-method comparison at control points spanning the parameter domain and the full range of the planar-surface error: Müller-Breslau (planar mechanism), rotational log-spiral upper bound (LS), and method of characteristics (SOK). $\delta = \varphi$ evaluated at $\varphi - 0.05^\circ$; '—' marks degenerate or physically meaningless closed-form values.

φ [°]	α [°]	β [°]	δ [°]	$K_{p,MB}$	$K_{p,LS}$	$K_{p,SOK}$	(MB-SOK)/SOK [%]	(LS-SOK)/SOK [%]
30	0	0	15	4.977	4.649	4.614	+7.8	+0.8
20	0	0	20	3.519	3.126	3.043	+15.7	+2.7
30	10	10	20	7.162	5.996	5.936	+20.6	+1.0
40	0	0	20	11.771	9.815	9.667	+21.8	+1.5
30	-10	0	20	9.663	7.066	6.896	+40.1	+2.5
30	0	0	30	10.066	6.920	6.543	+53.8	+5.8
50	0	0	25	49.025	27.217	26.388	+85.8	+3.1
40	0	0	40	91.704	19.981	18.112	+406	+10.3
40	-15	0	26.7	128.982	23.991	22.802	+466	+5.2
45	0	10	30	426.155	36.541	35.006	+1117	+4.4
45	0	0	45	$3.72 \cdot 10^6$	39.412	34.897	—	+12.9
50	0	0	50	—	89.716	77.808	—	+15.3
30	-20	-25	20	2.891	2.872	2.850	+1.5	+0.8

The kinematic bound exceeds the characteristics solution by 0.8–15% — a wider one-sided gap than the 0.2–2% observed in the active case, as expected for the passive rotational mechanism — but the ordering never inverts, and the Lancellotta bound encloses the solution from below wherever it applies. The decisive observation is method-independent: at $\varphi = 40^\circ$, $\alpha = \beta = 0$, $\delta = \varphi$, the log-spiral upper bound alone is 19.98 against the closed-form 91.70 — the planar solution exceeds a rigorous upper bound on the exact coefficient by a factor of 4.6, so its overestimation at high wall friction is established without any reference to the slip-line solver. At the anchor point $\varphi = 30^\circ$, $\delta \rightarrow \varphi$, the characteristics value 6.54 agrees with the Kérisel-Absi tables (≈ 6.55) to 0.2%, while the closed form gives 10.07.

The finite-element limit analysis programme provides the sharpest and most extensive test. All 660 analyses (330 lower and 330 upper bounds) converged; the bound gap UB – LB, relative to the bracket mean, had a median of 0.69%, a 95th percentile of 2.19% and a maximum of 8.73% (φ

$= 45^\circ, \delta = \varphi$). With a $\pm 1\%$ numerical tolerance on the bounds, the slip-line coefficient fell inside the FELA bracket in 305 of the 330 cases (92.4%), including 14 of the 17 combinations beyond the degeneracy locus — the region where the Müller-Breslau expression offers no prediction whatever, and where the rigorous bracket now confirms the slip-line coefficients directly (Figure 5). The 25 exceptions form two sharply defined families, mapped in Figure 7, and both admit a complete explanation.

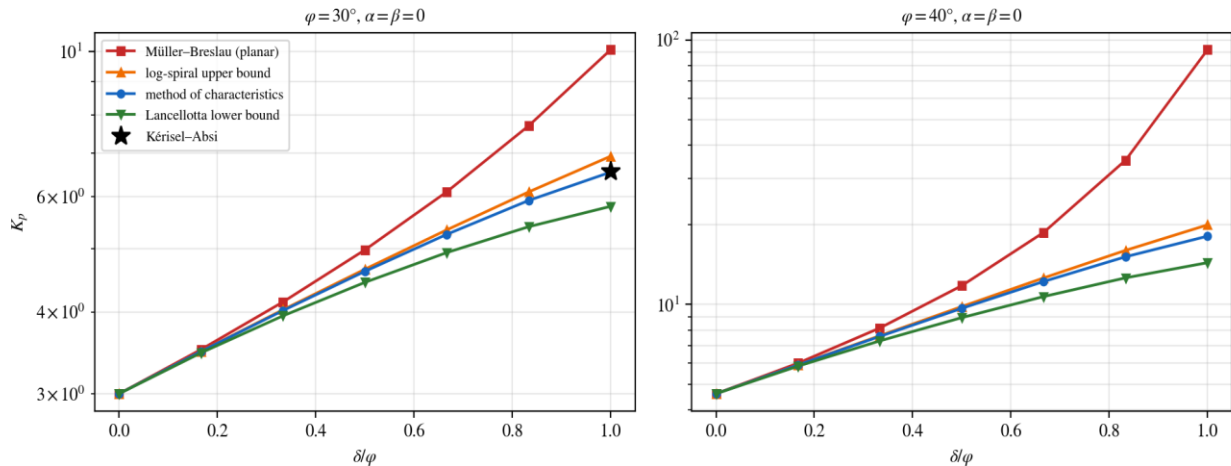


Figure 4. Wall-friction sweeps at $\alpha = \beta = 0$ for $\varphi = 30^\circ$ and $\varphi = 40^\circ$ (logarithmic scale): Müller-Breslau closed form, rotational log-spiral upper bound, method of characteristics, and Lancellotta lower bound; the star marks the Kérisel-Absi value at $\varphi = 30^\circ, \delta = \varphi$. The exact coefficient lies between the green and orange curves; the red curve leaves the bracket as δ grows.

The first family, F1 (21 cases), is the smooth wall inclined away from the backfill: $\delta = 0, \alpha = +10\dots+25^\circ$. There the planar Müller-Breslau mechanism is the exact solution — in all 21 cases the closed-form coefficient lies inside the FELA bracket, within 0.4% of its midpoint — while the present slip-line implementation (the stress-jump construction, branch B) is conservative by 1.2% ($\varphi = 20^\circ, \alpha = +15^\circ$) up to 8.1% ($\varphi = 45^\circ, \alpha = +25^\circ$), the deficit growing systematically with φ and α (Figure 6). This region is therefore not a limitation of the closed form but of the slip-line branch-B construction, and it qualifies Table 2 as noted there: in the $\delta = 0$ column at $\alpha > 0$ a part of the tabulated difference reflects slip-line conservatism rather than planar-surface error. The practical conclusions of this study, which are driven by the high-friction range, are unaffected.

The second family, F2 (4 cases), is the rough wall with strong overhang: $\delta = \varphi, \alpha = -25^\circ, \varphi \geq 30^\circ$. There the FELA collapse load of a translating rigid wall exceeds the slip-line value by 1.3–5.3%. Integration of the interface tractions shows the cause: under purely horizontal translation the interface cannot mobilise the full δ near the heel, where the pressure peaks, so the resultant is less inclined than assumed (integrated $\delta_{\text{mob}} \approx 33\text{--}37^\circ$ instead of 45° at $\varphi = 45^\circ$) and the horizontal component — hypersensitive to δ_{mob} when $|\alpha - \delta|$ approaches 70° (a change of 1° alters P_h by about 1.6%) — rises above the classical value. The classical coefficients themselves are confirmed independently in this region by the Kérisel-Absi tables; the discrepancy quantifies the kinematic

reachability of the classical boundary condition, not an error of either method.

Passive coefficient, rough wall ($\delta = \varphi$), $\beta = 0$ — slip-line vs FELA bounds vs Müller-Breslau vs Kérisel-Absi

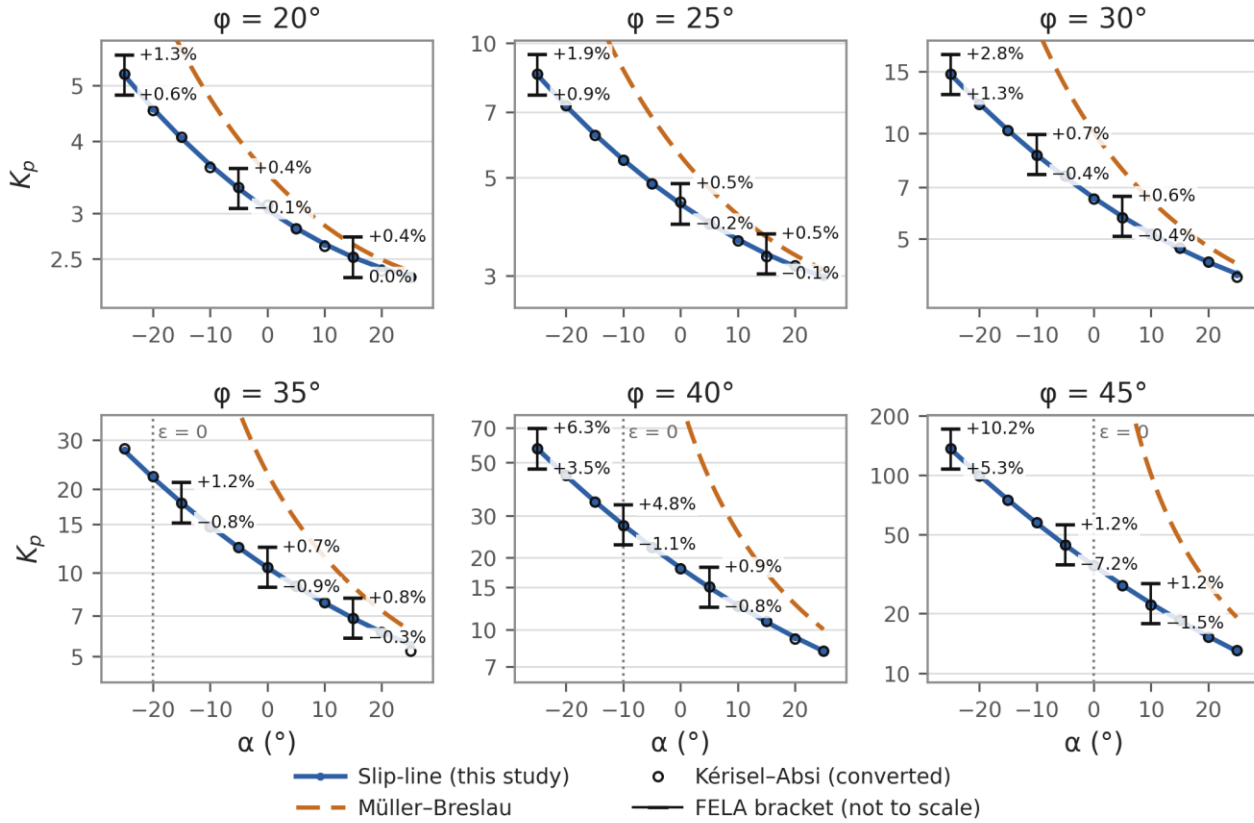


Figure 5. Rough wall ($\delta = \varphi$, $\beta = 0$): slip-line solution, rigorous FELA bracket (markers not to scale — the percentages give the actual deviations of the lower and upper bound from the slip-line value), Müller-Breslau closed form and converted Kérisel-Absi values (circles); the dotted vertical line marks the degeneracy locus $\varepsilon = 0$. Beyond the locus the closed form yields no prediction, while the slip-line coefficient remains finite and inside the bracket.

The comparison with the Kérisel-Absi tables [7] was accordingly extended from the single anchor point to the entire $\delta = \varphi$ branch. With the normalisation conversion $K_p = K_{p,K-A}/\cos^2\lambda$, $\lambda \equiv \alpha$ (their coefficient acts on the face length), the 66 tabulated combinations ($\varphi = 20\text{--}45^\circ$, $\alpha = -25\text{--}+25^\circ$, table $\delta/\varphi' = -1.00$, $\beta = 0$) agree with the slip-line coefficients with a median difference of -0.1% and within $\pm 2\%$ everywhere except one reading ($\varphi = 35^\circ$, $\alpha = +25^\circ$: -5.2% , at the two-significant-digit resolution of the table); at $\varphi = 45^\circ$ the agreement is within $\pm 1.5\%$ at all eleven inclinations (e.g. $\alpha = -25^\circ$: 136.4 against the converted 135.5; $\alpha = 0$: 35.0 against 34.9). The active-branch counterpart of the same comparison (90 tabulated combinations, $\delta/\varphi \in \{0, 2/3, 1\}$) agrees to a median of -0.2% . Two fully independent methods thus confirm the present solver on both limit states — in particular across the whole region where the Müller-Breslau passive expression is undefined.

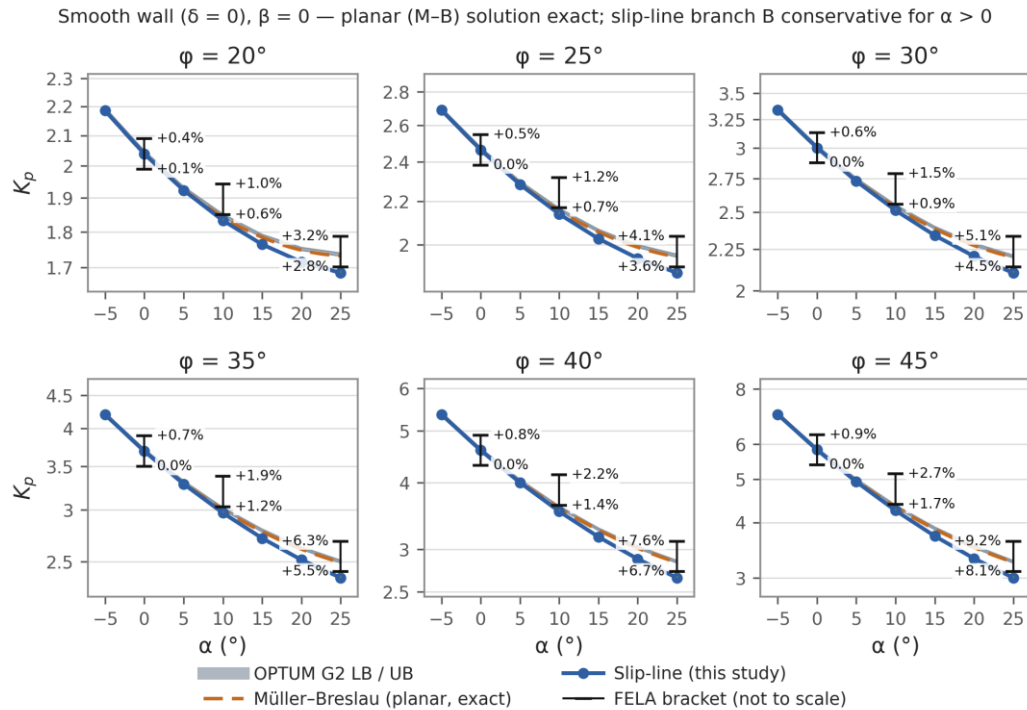


Figure 6. Smooth wall ($\delta = 0$, $\beta = 0$): the planar Müller-Breslau mechanism is exact — it lies inside the FELA band throughout — while the slip-line branch-B construction becomes conservative for $\alpha > 0$ (family F1): at $\alpha = +25^\circ$ the entire FELA bracket lies above the slip-line value. The view is restricted to $\alpha \geq -5^\circ$; for smaller α the solutions are indistinguishable at this scale. Bracket markers are not to scale; the percentages give the actual deviations from the slip-line value.

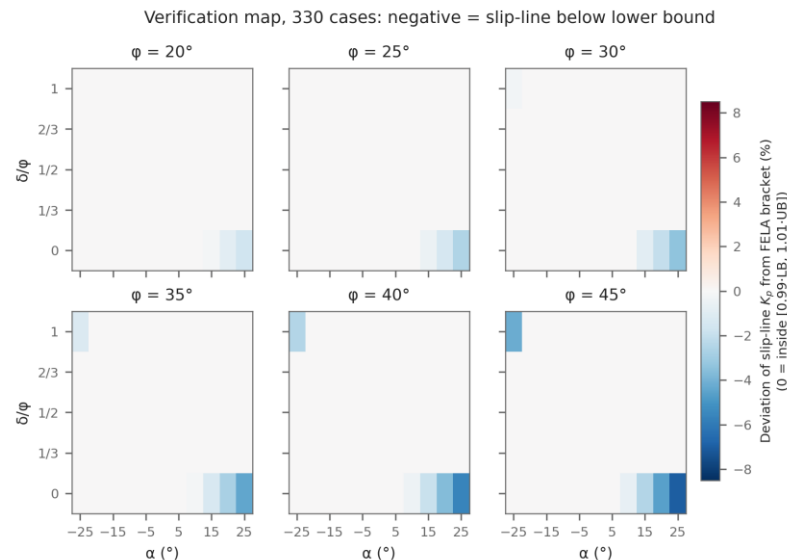


Figure 7. Verification map of all 330 FELA cases: deviation of the slip-line coefficient from the rigorous bracket (0 = inside the bracket at $\pm 1\%$ tolerance). The deviation families F1 (slip-line branch-B conservatism, $\delta = 0$, $\alpha \geq +10^\circ$) and F2 (incomplete interface mobilisation, $\delta = \phi$, $\alpha = -25^\circ$) are the only non-zero regions.

Behaviour beyond the degeneracy locus: sensitivity to the sub-heel boundary

Beyond the degeneracy locus $\varphi + \delta + \beta - \alpha = 90^\circ$, the collapse mechanism no longer remains confined between the wall face and the free surface: it dips below the plane of the wall heel and, if soil is present there, engages material behind the plane of the wall (Figure 8). For the extreme cell $\varphi = 45^\circ$, $\delta = \varphi$, $\alpha = -25^\circ$ ($\varepsilon = -25^\circ$) the horizontal collapse load of the translating wall is 490–512 kN/m when the region below the heel is idealised as a smooth rigid boundary — the closest analogue of the classical assumption — but 876–924 kN/m when a soil continuum is modelled below and behind the heel: almost twice the classical $P_h = 464.6$ kN/m computed from the slip-line coefficient, even though the magnitude of the total resultant remains close to the classical value. In this region the passive resistance therefore ceases to be a property of $(\varphi, \alpha, \beta, \delta)$ alone: it depends decisively on conditions below the heel that the classical framework leaves unspecified. Together with the incomplete wall-friction mobilisation of family F2, this constitutes a practical limitation of the classical passive-pressure framework that is distinct from, and additional to, the formal degeneracy of the Müller-Breslau expression. These observations delimit, rather than resolve, the sub-heel problem: the present study documents the sensitivity and identifies its mechanism, while its systematic quantification — the roles of wall thickness, base contact and the shear transmissible across the wall-soil interface — is deliberately left outside the scope of this paper and is the subject of a dedicated follow-up study. All coefficients reported here for the degenerate zone therefore refer to the classical configuration, with a rigid, smooth boundary at heel level.

Collapse mechanism extent (UB adapted-mesh refinement, element size < 0.2 m) — $\varphi = 45^\circ$, $\delta = \varphi$, $\alpha = -25^\circ$

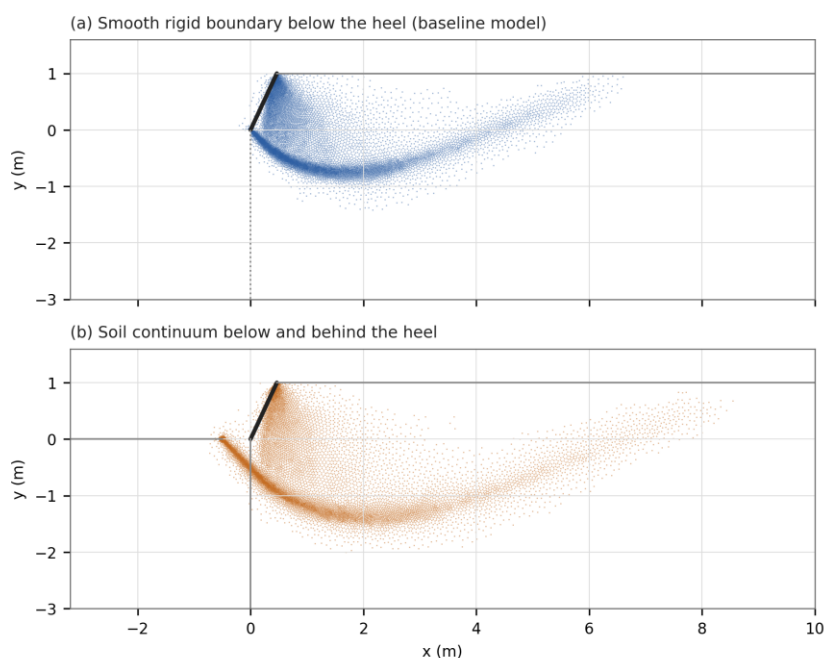


Figure 8. Collapse-mechanism extent for $\varphi = 45^\circ$, $\delta = \varphi$, $\alpha = -25^\circ$, reconstructed from the adapted-mesh refinement: (a) smooth rigid boundary below the heel (the analogue of the classical assumption); (b) soil continuum below and behind the heel. In case (b) the mechanism extends about 1.6 m behind the plane of the wall ($H = 1$ m) and the horizontal collapse load nearly doubles.

Applicability maps

The findings are condensed into the applicability maps of Figure 9, which classify the (β, δ) plane for each wall inclination into zone A ($|\Delta| < 2\%$), zone B (2–5%), zone C ($\geq 5\%$), the degenerate zone D without a closed-form prediction, and the excluded region $|\beta| \geq \varphi$, in which a cohesionless slope would exceed its natural angle of repose. The proportions contrast sharply with the active-case maps [25]. Over the whole grid, zone A occupies 19.5% of the physically admissible cells, zone B 10.4%, zone C 52.7% and zone D 17.4%. The safe zone is largest at low friction angle and positive wall inclination — 53% of the plane at $\varphi = 20^\circ, \alpha = +25^\circ$ — and shrinks monotonically with growing φ and decreasing α , to 9.5% at $\varphi = 50^\circ, \alpha = 0^\circ$ and 2.4% at $\varphi = 50^\circ, \alpha = -25^\circ$ (Figure 10b); in the active case the corresponding shares for near-vertical walls were 68–84%. The practical reading is direct: the passive Müller-Breslau formula is trustworthy only in the low-friction, low-slope corner of the parameter space, far from the degeneracy locus; for the parameter ranges in which passive resistance actually matters in design — dense soils, rough interfaces — a curved-surface solution is required, and for a conservative closed-form alternative the Lancellotta bound (6) is available at no computational cost.

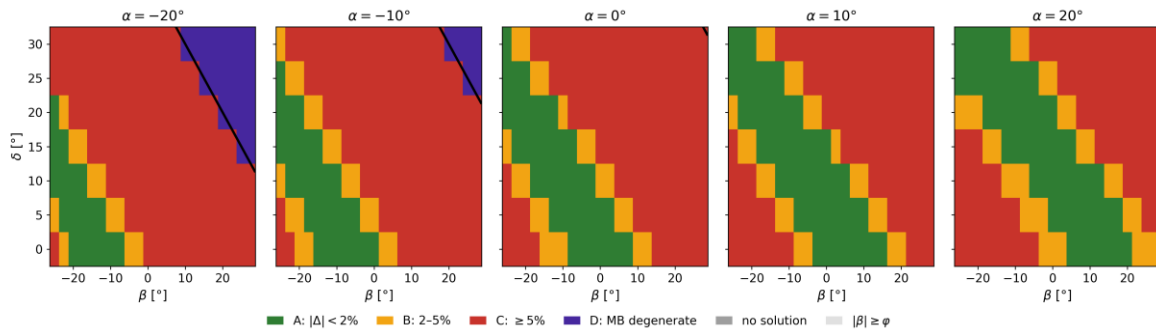


Figure 9. Applicability map of the passive Müller-Breslau formula (example shown for $\varphi = 30^\circ$): zone A ($|\Delta| < 2\%$), zone B (2–5%), zone C ($\geq 5\%$), zone D (degenerate — no closed-form prediction), grey — unresolved cells, light grey — inadmissible $|\beta| \geq \varphi$. The black line is the degeneracy locus $\varphi + \delta + \beta - \alpha = 90^\circ$.

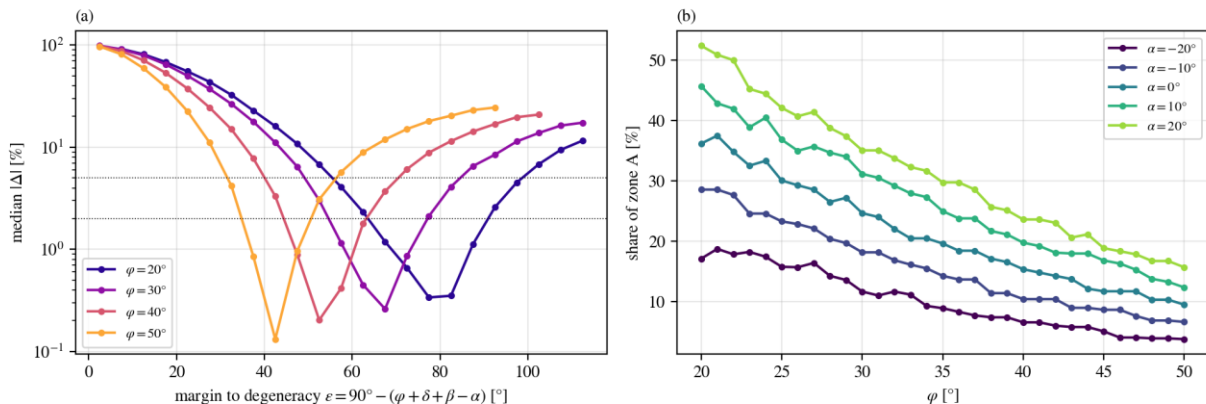


Figure 10. Summary over the friction-angle range: (a) median planar-surface error $|\Delta|$ as a function of the margin ε to the degeneracy locus, collapsing the α, β and δ dependences onto one curve per φ (logarithmic scale; dotted lines at 2% and 5%); (b) share of the safe-application zone

A as a function of φ for each wall inclination α . At very large margins — reachable only for low δ , strongly descending backfill and $\alpha > 0$ — the median difference grows again, now on the conservative side (the branch-B region qualified at Table 2).

The results admit a transparent interpretation in the framework of the limit theorems [8]. The Coulomb wedge is the simplest kinematic mechanism for the passive problem, and its exact evaluation — the Müller-Breslau closed form — is consequently a mechanism-based estimate approaching the true resistance from above; the rotational log-spiral mechanism is a better member of the same kinematic family and lies between; the Lancellotta stress field approaches from below; and the method of characteristics, constructing a complete stress solution, falls inside the bracket. The ordering $LB \leq SOK \leq LS \leq MB$ observed at every verified point of Tables 1 and 3 is exactly the prediction of the theorems, with every inequality reversed relative to the active case. Within this picture, the machine-precision agreement between the closed form and the constrained numerical wedge is not a mechanical result but an essential verification: it certifies that the entire difference reported in Tables 2–3 belongs to the failure-surface geometry, and not to the algebra or numerics of the wedge model.

The governing role of the margin ε has a simple geometric explanation. The admissible fan of trial planes is $(\beta, \rho_{\text{sing}})$, and its angular width is precisely ε . When the fan is wide, the planar optimum sits far from the singular direction, the true slip surface is gently curved, and the wedge estimate is adequate. As the fan narrows — whether by increasing δ or β , by decreasing α , or by increasing φ , all of which enter ε with equal weight — the critical plane is displaced toward the singular direction, along which the wall and slip-plane reactions are parallel and the planar model can balance the wedge only at rapidly growing force magnitudes; the true mechanism escapes this geometric trap by curving, which the planar surface cannot do, and the error grows without bound as $\varepsilon \rightarrow 0$. This also explains why no individual parameter dominates: the error is a function of the aggregate $\varphi + \delta + \beta - \alpha$ far more than of its components. In the active case, the corresponding fan is wide throughout the practical domain, the singular direction is reached only at the boundary of the practical domain, and the error structure is instead governed by the wall inclination [25]; the two papers thus identify two different geometric mechanisms by which the same planar idealization fails.

The practical implications are direct, and they are more severe than in the active case because passive resistance is a stabilizing quantity: overestimating it reduces real safety margins wherever passive earth pressure is relied upon — embedded retaining walls below the excavation level, anchor blocks and deadman anchors, thrust blocks, bridge abutments, and the toe resistance of gravity structures. In the parameter ranges typical of such applications (dense granular soils, $\varphi \geq 35^\circ$, cast-against or rough interfaces with $\delta \geq 2\varphi/3$), Table 2 shows median overestimations of 20–50% and the maps of Figure 9 place virtually the whole region in zones C and D. Three graded recommendations follow. First, the closed form may be retained where the margin ε exceeds roughly $45\text{--}50^\circ$, which in practice confines its use to low friction angles and low wall friction, with unfavourable geometries (strongly negative α , steep backfill) excluded by an explicit check of the margin ε . Second, wherever ε is smaller, a curved-surface solution — the Kérisel-Absi tables [7], a

characteristics or finite-element limit analysis computation [13], or the log-spiral mechanism — should replace it; the present maps quantify the error committed if it is not. Third, when a closed-form expression is wanted, the Lancellotta bound (6) is itself one; it is rigorous, and it errs on the safe side by construction; for the vertical rough wall it reproduces the exact coefficient within 11–35% from below over the whole friction range examined here, against unbounded overestimation from above by the wedge formula. The degenerate zone D deserves separate emphasis: it is not an exotic corner but 17.4% of the physical grid; it includes routinely specified combinations such as $\varphi = 50^\circ$, $\delta = \varphi$ on a vertical wall, and within it the formula produces either absurdly large values just before the locus or nothing at all beyond it — while the exact coefficient is finite, computable, and in this study bracketed to within tens of percent.

Two numerical by-products deserve emphasis because they affect anyone re-implementing the passive wedge optimization, including in teaching. First, since the objective function of the minimization, $K_p(\rho)$ in (4), is defined only on the fan $(\beta, \rho_{\text{sing}})$ and grows without bound at both ends, bracketing scalar optimizers silently fail on the infinite plateau; a dense constrained scan is required. Second, the search interval must be anchored at β , not at zero: for $\beta < 0$ near the locus, the critical plane dips below the horizontal, and a search over $\rho > 0$ returns values grossly in error (232% in the example quoted in the Methodology) while appearing perfectly converged. Both failure modes produce plausible-looking coefficients and neither condition is stated in the standard derivations [2], [3]. The margin ε , computable in one line, is the natural guard: implementations should refuse, or at least flag, evaluations at small ε — which is precisely where naive optimizations misbehave and where the formula, even correctly evaluated, is furthest from the truth.

The study is limited to a dry, homogeneous, cohesionless backfill under static conditions, to resultant forces rather than pressure distributions, and to the associated flow rule implicit in both limit analysis and the method of characteristics; for the strongly non-associated behaviour of dense sands the rigorous coefficients themselves are known to decrease further, so the planar-surface error quantified here is, if anything, a lower estimate of the total unconservatism. The displacement required to mobilize full passive resistance — many times that of the active state — is outside the scope of limit theory and must be checked separately in serviceability terms. The extension to cohesive backfill via the theorem of corresponding states is exact only for smooth walls, so the present error maps cannot simply be transformed to c - φ soil; the seismic extension through the Mononobe-Okabe framework, in which the passive coefficient is the Müller-Breslau solution under rotated gravity, suggests that the ε -governed structure identified here transfers directly, with the seismic rotation acting as an additional erosion of the margin — both extensions, together with the surcharge case needed for direct comparison with Sokolovski's own tables, are left for future work. Finally, beyond the degeneracy locus the classical problem itself becomes incompletely posed: the finite-element analyses show the collapse mechanism extending below the wall heel and behind the plane of the wall, so the mobilisable resistance depends on sub-heel conditions that no coefficient of $(\varphi, \alpha, \beta, \delta)$ can capture; the coefficients reported here for that

region are those of the classical boundary-value problem, in which the collapse mechanism is implicitly confined above heel level; in the finite-element verification this configuration was approximated by a rigid, smooth boundary at the heel plane.

Conclusion

This paper mapped the validity domain of the Müller-Breslau passive earth pressure coefficient and quantified the error of its planar failure-surface assumption over the parameter space $\varphi = 20\text{--}50^\circ$, $|\alpha| \leq 25^\circ$, $-25^\circ \leq \beta < \varphi$, $0 \leq \delta \leq \varphi$. The main findings are as follows.

1. Direct numerical minimization of the Coulomb trial wedge, with the admissibility conditions $\beta < \rho < \rho_{\text{sing}} = 90^\circ - \varphi - \delta + \alpha$ imposed, reproduces the closed form to machine precision (below $10^{-10}\%$) over all 45 948 parameter combinations at which the formula is defined; the difference between the Müller-Breslau solution and the method of characteristics is therefore a pure measure of the planar failure-surface error.

2. The degeneracy of the closed form is characterized analytically: the radicand of the formula reaches unity exactly on the locus $\varphi + \delta + \beta - \alpha = 90^\circ$, on which the singular direction coincides with the backfill surface and the admissible fan of trial planes closes — the divergence of the formula and the disappearance of the wedge mechanism are the same event, with the asymptotic law $K_p \propto \varepsilon^{-2}$ in the margin $\varepsilon = 90^\circ - (\varphi + \delta + \beta - \alpha)$. Beyond the locus (17.4% of the physical grid), the formula yields no prediction, although the exact resistance is finite: the characteristics solver converged in 9 645 of 9 646 such cells, with coefficients up to 5 300.

3. The planar-surface error lies on the unsafe side in 99.6% of the grid and is governed primarily by the margin ε rather than by any individual parameter (Spearman -0.83 against at most 0.48): the median overestimation grows from about 2% for $\varepsilon > 50^\circ$ through 13.5% at $\varepsilon = 30\text{--}40^\circ$ and 52% at $\varepsilon = 15\text{--}20^\circ$ to divergence at the locus; at full wall friction $\delta = \varphi$ the median overestimation factor is 1.51, reaching 1.16, 1.54 and 5.06 at $\varphi = 20, 30, 40^\circ$ on the vertical wall and unbounded values beyond.

4. The characteristics solution is enclosed at every verified point in the rigorous sandwich $K_{p, \text{LB}} \leq K_{p, \text{SOK}} \leq K_{p, \text{LS}} \leq K_{p, \text{MB}}$ formed by the Lancellotta lower bound and the rotational log-spiral upper bound, reproduces the Rankine limit to 0.04%, respects all 35 upper-bound coefficients of Soubra, and agrees with the Kérisel-Absi anchor value at $\varphi = 30^\circ$, $\delta \rightarrow \varphi$ to 0.2%. The overestimation by the closed form is established independently of the solver: at $\varphi = 40^\circ$, $\delta = \varphi$ the planar value exceeds the rigorous upper bound alone by the factor 4.6. Finite-element limit analysis extends this verification: over a 330-case matrix the slip-line coefficient falls inside the rigorous LB–UB bracket in 92.4% of cases — including 14 of the 17 combinations beyond the degeneracy locus — and the residual deviations are confined to two explained families: conservatism of the slip-line stress-jump construction for the smooth wall inclined away from the backfill (where the planar solution is exact), and incomplete interface mobilisation for the strongly overhanging rough wall. Across the whole $\delta = \varphi$ branch, the converted Kérisel-Absi values agree with the slip-line coefficients to a median of 0.1%. Beyond the locus, additionally, the finite-element analyses show the collapse mechanism extending below the wall heel, so the mobilisable resistance becomes sensitive to sub-heel conditions outside the classical framework — a limitation documented here and left to a dedicated follow-up study.

5. Applicability maps classify the parameter space into a safe-application zone ($|\Delta| < 2\%$; 19.5% of the physical grid overall, from 53% at $\varphi = 20^\circ$, $\alpha = +25^\circ$ down to 2.4% at $\varphi = 50^\circ$, $\alpha = -25^\circ$), a caution zone (10.4%), a significant-error zone (52.7%) and the degenerate zone (17.4%).

For design practice, the passive Müller-Breslau formula should be restricted to the large-margin region $\varepsilon \gtrsim 45\text{--}50^\circ$; elsewhere a curved-surface solution is required, and where a closed form is wanted, the Lancellotta lower bound offers a rigorous conservative substitute. The interactive applications accompanying this paper allow the complete parameter space to be explored directly.

The contribution of the authors

Aleksej Aniskin: conceptualization, methodology, software, validation, formal analysis, visualization, writing — original draft.

Khrystyna Moskalova: writing — review and editing.

Matija Orešković: writing — review and editing.

Andjelko Crnoja: writing — review and editing.

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Availability statement

The parametric-study data (56 364 cells), the validation data, and the computational scripts supporting the conclusions of this study are available from the corresponding author upon reasonable request. The interactive applications are archived on Zenodo: the Sokolovski K_a/K_p slip-line solver (<https://doi.org/10.5281/zenodo.22011079>) and the interactive 3D visualization of the Müller–Breslau coefficients (<https://doi.org/10.5281/zenodo.21839813>).

Supplementary materials

Interactive web applications for the exploration of the passive (and active) earth pressure coefficients over the full parameter space are available on Zenodo: the interactive 3D visualization of the Müller–Breslau coefficients K_a/K_p with a synchronized retaining-wall cross-section

(<https://doi.org/10.5281/zenodo.21839813>) and the Sokolovski K_a/K_p slip-line solver with Müller–Breslau and Coulomb-wedge comparison (<https://doi.org/10.5281/zenodo.22011079>).

Use of Artificial Intelligence (AI)

Grammarly AI V1.2.290

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